Some Exact Solutions of the NLS Equation for Deep-water Waves

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ABSTRACT

Whitham derived the nonlinear Schrödinger equation from the most general form of the nonlinear wave dispersion relation. The NLS equation was analyzed from the second-order Stokes’ nonlinear deep-water wave dispersion relation. The exact solutions of the nonlinear wave were obtained by using modified mapping method. As this specialization of this equation, the solutions were different from that had been obtained and contained the Debnath’s solutions. From the form and the character of the solutions, the relation of the solutions was analyzed and these can be classified to three kinds.

KEY WORDS: Nonlinear Schrödinger equation; dispersion relation; solitary wave; Jacobi elliptic functions.

INTRODUCTION

The nonlinear Schrödinger equation is usually derived from the basic governing equations using the multiple scales method. The nonlinear Schrödinger equation was first obtained by Zakharov(1968) from the basic governing equations. It’s the first equation describing evolution of nonlinear gravity wave groups. And many different forms of NLS equations such as Davey–Stewartson equation (DSE) (Davey & Stewartson 1974), the modified nonlinear Schrödinger equation (MNLS) (Dysthe,1979) and the broader bandwidth nonlinear Schrödinger equation(BMNLS) (Trulsen,1996) were developed. Based on these equations, the instability of wave trains was studied (Hasimoto,1972; Yuen,1975; Stiassnie,1987; Canney,2006), and also the deep-water wave evolution was simulated by solving these equations with numerical methods (Lo,1985; Osborne, 2000; Trulsen,2001; Onorato,2001). Studying the exact solution of nonlinear Schrödinger equation has important theoretical and practical significance to the theory of water wave. Recently, a lot of methods were used to solve nonlinear partial differential equation. Such as, inverse scattering (Gardner,1967), the homogeneous balance method (Wang,1995), separation of variables (Zhang,2004), sin-cosine function expansion method(Yan,1996), tanh function expansion method (Li,2004), Jacobi elliptic function expansion method (Liu,2002) and modified mapping method (Peng,2005). And there are some different exact solutions of the NLS equation had been got by using different mathematic methods(Hui,1979; Peregrine,1983; Ma,1979; Dysthe,1999; Abourabia,2010).

Whitham (1974) derived wave propagation equation with the form of the nonlinear Schrödinger equation from the most general nonlinear dispersive relation. Debnath(1994) solved this equation and obtained two exact solutions. In this paper we solve the NLS equation with the second-order Stokes’ nonlinear deep-water wave dispersion relation and get some exact solutions by using the modified mapping method.

THE DEEP WATER WAVE NONLINEAR SCHRODINGER EQUATION

The nonlinear Schrödinger equation

\[ i\phi_t + \alpha \phi_{xx} + \beta \phi |\phi|^2 = 0 \]  \hspace{1cm} (1)

where \( i = \sqrt{-1} \), \( \phi \) is function, \( \alpha \), \( \beta \) are coefficients, \( x \) is spatial coordinate.

The nonlinear wave dispersive relation in the most general form,

\[ \omega = \omega(k, a^2) \]  \hspace{1cm} (2)

Where \( \omega \) is angular velocity, \( k \) is wave number, \( a \) is the amplitude of the water wave.

Whitham derived from Eq. 2 and got

\[ i(a_t + a_0 a_x) + \frac{1}{2} a_0 a_{xx} + \beta |a|^2 a = 0 \]  \hspace{1cm} (3)

Where \( a_0 = \omega(k, a^2) \) \( k=k_0, |a|^2=0 \), \( a_0 = \left( \frac{\partial \omega}{\partial k} \right)_{k=k_0, |a|^2=0} \), \( \beta = \left( \frac{\partial \omega}{\partial |a|^2} \right)_{k=k_0, |a|^2=0} \), \( k_0 \) is the wave number of the carrier wave in the wave trains.

Transformed Eq. 3 with the variables \( x^* = x - a_0 t^* \), \( t^* = t \), (3) can be rewritten as the form of the NLS equation, dropping the asterisks,

\[ ia_t + \frac{1}{2} a_0 a_{xx} + \beta |a|^2 a = 0 \]  \hspace{1cm} (4)

The dispersive relation of the second-order Stokes wave in deep water

\[ \omega^2 = gk(1 + k^2 a^2) \]  \hspace{1cm} (5)