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Analytical Solution for Long Waves on Axis-Symmetric Topographies

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ABSTRACT

In this study, we develop analytical solutions for long waves propagating over several types of axis-symmetric topographies where the water depth varies in an arbitrary power of radial distance. The first type is a cylindrical island mounted on a shoal. The second type is a circular island. To get the solution, the methods of separation of variables, Taylor expansion and Frobenius series are used. The developed analytical solutions are validated by comparing with previously developed analytical solutions. We also investigate various cases with different incident wave periods, radii of the shoal, and the powers of radial distance.

KEY WORDS: Analytical solution; long wave; axis-symmetric topography; Frobenius series.

INTRODUCTION

Several researchers obtained analytical solutions for long waves on axis-symmetric topographies such as a circular cylindrical island mounted on a paraboloidal shoal (Homma, 1950), a conical island (Zhang and Zhu, 1994) a circular cylindrical island mounted on a conical shoal (Zhu and Zhang, 1996), and a circular paraboloidal pit (Suh et al., 2005). However, most solutions are limited to the cases that water depth is proportional to the 1st or 2nd power of radial distance. In the real bottom topography, however, the water depth's power of the radius would be less than one. Bruun (1954) argued that water depth is proportional to two-thirds power of offshore distance from the shoreline in an equilibrium beach.

In this study, we developed analytical solutions for long waves on several types of axis-symmetric topographies where the water depth varies in an arbitrary power of radial distance; a cylindrical island mounted on a shoal and a circular island. For the case of a cylindrical island, Yu and Zhang's (2003) solution was restricted to the case that the vertex of the topography was located at the mean water level while the vertex considered in this study may be located at or below the mean water level. In the case of a circular island, Zhang and Zhu's (1994) solution was limited only to a conical island.

DEVELOPMENT OF MODEL EQUATIONS

For linear regular waves of incompressible and inviscid water, the mild-slope equation can be obtained as

$$\nabla \cdot (CC_g \nabla \eta) + \omega^2 \frac{C_g}{C}\eta = 0 \tag{1}$$

where ∇ is the horizontal gradient operator, η is the complex water surface elevation, C is the phase speed, C_g is the group velocity, and ω is the angular frequency. When the waves propagate in shallow water region, Eq. 1 can be written in a polar coordinate as

$$h\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}\right) + \frac{dh}{dr}\frac{\partial \eta}{\partial r} + \frac{\omega^2}{g}\eta = 0 \tag{2}$$

By separating variables as $\eta(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta)$, Eq. 2 can be split into the following two equations

$$h\frac{d^2 R_n}{dr^2} + \left(\frac{h}{r} + \frac{dh}{dr}\right)\frac{dR_n}{dr} + \left(\frac{\omega^2}{g} - \frac{n^2 h}{r^2}\right)R_n = 0 \tag{3}$$

$$\Theta_n(\theta) = C_{1n}\cos n\theta + C_{2n}\sin n\theta \tag{4}$$

where $R_n(r)$ is the function of r corresponding to each eigenvalue n, and C_{1n} and C_{2n} are constants.

SOLUTIONS FOR A CYLINDRICAL ISLAND MOUNTED ON A SHOAL

Derivation

Firstly, we developed analytical solutions for waves around a cylindrical island mounted on a shoal. The water depth on the shoal varies in proportion to an arbitrary power of radial distance. Fig. 1 shows the computation domain for waves around the cylindrical island on the shoal. In the figure, h_0 and h_1 are the water depths at the island wall and the shoal toe, respectively, and, r_0 and r_1 are the corresponding radial distances from the island center. The water depth on the shoal is expressed by