Mooring Response of a Floating Offshore Wind Turbine in Nonlinear Irregular Waves

Christof Wehmeyer
Rambøll Offshore Wind
Esbjerg, Denmark

Jørgen Hvenekær Rasmussen
DONG Energy
Fredericia, Denmark

The current work focuses on the mooring loads obtained by a numerical model of a floating offshore wind turbine (FOWT) foundation, where the substructure is a three-legged Tension Leg Platform (TLP) and the influences of a flexible topside are considered. Representatively, the behaviour of one tendon is investigated. The FOWT is represented by a hybrid model that combines linear inertial excitation forces, obtained by potential flow, with nonlinear hydrodynamic viscous drag forces. The latter are obtained from the relative velocities between the linear and nonlinear sea state kinematics and the floater. The nonlinear sea state surface elevation and kinematics are generated by a second-order wave model including all sub- and super-harmonics, with and without an embedded Stream-function wave. The numerical responses of the bow tendon are compared to respective observations from model tests. The maximum bow tendon loads are overpredicted by the numerical model by 34% and 32% for the nonlinear and linear sea state generation, respectively. The embedded wave approach results in an overprediction of 37% to 23%, depending on the different crest front steepness values of the single wave, which is embedded in the nonlinear irregular background sea state.

INTRODUCTION

The focus of location choices for the possible deployment of floating offshore wind turbine (FOWT) foundations has shifted lately. Until recently, the industry anticipated that the majority of the market will be in deep water areas outside of Europe, in water depths > 100 meters. However, a number of research and prototype projects have been initiated to realize installation in the deeper areas of the European continental shelf sea, i.e., in intermediate water depths ≤ 60 meters. A recent example is the UK Crown Estate’s demonstrator project, which targets a commercially viable FOWT solution by 2020, i.e., a Tension Leg Platform (TLP) design in approximately 60 meters water depth and with significant wave heights of more than 10 meters. At such water depths, a linear wave assumption insufficiently describes realistic wave shapes, wave kinematics, and thereby wave force in the design conditions, which differentiates FOWTs from earlier floater applications. Additionally, FOWTs are commonly composed of slender parts at the free surface and voluminous submerged parts. The submerged parts are prone to excitation by diffractive wave forces in large and small sea states. However, slender surface piercing parts are excited by hydrodynamic viscous drag in large sea states. Another important aspect is that traditionally floating structures have been considered rigid; the influence of their structural dynamics on the global system behaviour was negligible. However, the high flexibility of FOWT topsides including the rotor nacelle assembly (RNA) and tower changes the global system’s response, especially for TLPs (Matha, 2009). Therefore, the current work attempts to assess if a key response of a FOWT located in intermediate water depths can be reasonably well represented by a hybrid model, which considers a dynamically sensitive topside and combines linear radiation damping, linear inertial excitation, and quadratic viscous excitation from linear and nonlinear waves.

The second-order inertial effects caused by the sum and difference frequency interaction of the wave on FOWT TLP structures, e.g., as discussed in Bachynski and Moan (2013) and Roald et al. (2013), are not included in the current stage. The challenge is that the first-order hydrodynamic coefficients can be obtained solely from the structure’s geometry, i.e., the first-order wave forces are derived based on the assumption of a stationary floating body; however, the second-order wave forces are derived based on the assumption that the body is allowed to move and respond to the first-order wave forces. This implies that the first-order body motions are needed for the computation of the second-order fluid forces. For a FOWT TLP in particular, the rigid body assumption considered in the potential flow theory does not hold. This was first highlighted by Matha (2009) and later supported by physical model tests and respective numerical representations of natural frequencies for different topside flexibilities in Wehmeyer et al. (2013). Further work on this is ongoing; however, the current stage of the numerical model includes only higher-order viscous terms.

Recently, several efforts have been undertaken to investigate the effect of higher-order wave impact on offshore wind turbines. Agarwal and Manuel (2010) investigated the effect of a second-order irregular wave model on a bottom fixed structure, i.e., a monopile. Wheeler stretching was applied to determine the kinematics up to the free surface. Unsurprisingly, they showed that the second-order model predicted higher loads than a first-order model. Morison’s approach was applied to determine the wave excitation force on a 6-meter monopile in 20 meters water depth, which supported a 5MW reference turbine. An increase of 2% and 18% of the ten-minute maximum overturning moment at the mud line was found for significant wave heights of $H_m = 5.5$ meters and $H_m = 7.5$ meters,
respective. On the basis of this information, it can be concluded that the maximum fore-aft tower bending moment was due to the viscous drag. The maximum values occurred simultaneously with the wave crest passing the structure. It is known that second-order irregular wave models predict more realistic crest height distributions (Forristall, 2000), which means higher individual wave crests and consequently more realistic and higher viscous contributions above the still water level, especially for large sea states. It is therefore assumed that accurate viscous force contributions are an important part of the incident wave excitation for the investigated FOWT and should be considered accordingly.

**METHODS**

The DNV-OS-J103 guideline (2013) recommends two approaches for modelling the ultimate limit state (ULS) design conditions in shallow water or in conditions where waves become steep:

(A) Second-order irregular waves

(B) Embedding a Stream-function wave into an irregular Airy wave model

The overall purpose of the current work is to obtain insight into the applicability of options (A) and (B) in terms of FOWT behaviour in ULS conditions. The current work, however, modifies option B and embeds the Stream-function wave into a second-order nonlinear irregular wave in order to better match the overall crest distribution, as outlined above. The general purpose of embedding a particular wave is to control the largest wave in a design process, in which a defined maximum wave needs to be considered. First-order irregular waves are used for the sake of comparison.

While it is known that both approaches, strictly speaking, violate the linear theory assumption, it is widely accepted that the linear equation of motion for floating bodies can be extended by mildly nonlinear terms and can be practically applied to a wide range of engineering challenges. In the current work the applicability is evaluated, comparing numerical results to measurements of a key parameter, i.e., the loads experienced by the bow mooring of the FOWT TLP, obtained by a physical model testing campaign (Wehmeyer et al., 2013). The numerical response quality of option (A) is evaluated by spectral comparisons and by the probability of exceedance. The numerical response quality of option (B) is evaluated by the running average results of a consecutive number of exceedance. The numerical response quality of option (B) is determined by comparing the applied forces to the measured forces at the bow tendon, the maximum tower bending moment at the tower bottom, and the maximum wave force acting on the tower.

While the wind loads on the tower and the wind loads on the blades during fault are important in the design of offshore wind turbines, the physical model and the numerical model neglect the wind loads. The primary focus of the current work is to describe and understand the responses due to the hydrodynamic excitation. The company of Morison’s model (Jonkman et al., 2009) is evaluated, comparing numerical results to measurements of a physical model testing campaign (Wehmeyer et al., 2013). The numerical response quality of option (A) is evaluated by spectral comparisons and by the probability of exceedance. The numerical response quality of option (B) is evaluated by the running average results of a consecutive number of exceedance. The numerical response quality of option (B) is determined by comparing the applied forces to the measured forces at the bow tendon, the maximum tower bending moment at the tower bottom, and the maximum wave force acting on the tower.

Numerical Model

The numerical model is an integrated five degrees of freedom (5DOF) model of a rigid body floating substructure connected to a single linear elastic Bernoulli beam element, which represents the flexibility of the wind turbine tower and includes a lumped mass on top that represents the RNA. Figure 1 shows the numerical representation of the investigated system, which illustrates the flexural response of the indicated degrees of freedom by its indices. The shaded area indicates the flexible topside.

The assumed rigid substructure and the flexible topside are coupled through their shared degrees of freedom. The idealized structure has lateral symmetry in the x-z plane, which consequently means that surge, heave, and pitch are decoupled from sway, roll, and yaw, and hence a 2D system is assumed to be sufficient. The structural masses, the tower’s elasticity, and the structural damping are determined according to the physical model. The hydrodynamic coefficients for the added mass, hydrostatic stiffness, linear diffraction force, and linear radiation damping for the rigid body substructure are determined by use of the commercial potential theory solver, ANSYS AQWA (2013). The model includes the first-order inertial excitation forces (Froude-Krylov and diffraction) and the viscous hydrodynamic drag excitation force (the drag part of Morison’s equation), applying the relative velocities between the structure and the wave particle velocities up to the free surface. The radiation damping force is approximated by a state-space representation. The respective A, B, C, and D matrices of the fluid memory effect and the added mass at infinity are determined by a direct frequency domain identification procedure that applies the MSS FDI MATLAB toolbox (Perez and Fossen, 2009). The numerical model is consequently a time domain solver of the equation of motion, as given by Eq. 1:

\[
(M_{\text{tower}} + M_j + m h_0^2) \ddot{x}_i + \int_0^t C_{ij}^{\text{rad}} (t - \tau) \dot{x}_i (\tau) d\tau + (K_{ij}^{\text{hyd}} + K_{ij}^{\text{lower}}) x_i
\]

**(Fig. 1 Left: Numerical representation of investigated FOWT TLP, showing incident wave direction with blue arrow. Right: 3D view of FOWT TLP, defining the bow tendon and the incident wave direction.)**
\[ + F_{\text{Mooring}}^{\text{inertial}} + C_{ij}^{\text{tower}} \dot{x}_i = F_{i}^{\text{Exc}_\text{inertial}} + F_{i}^{\text{Exc}_\text{viscous drag}} \]

The left-hand side describes the body motion by:
- \((M_{\text{tower}} + M_{ij} + m h_i^n)\) Combined mass matrix, consisting of the topside mass matrix, the substructure mass matrix, and the constant infinite frequency added mass
- \(\int_{0}^{\tau} C_{ij}^{\text{hyd}} (t - \tau) \dot{x}_i (\tau) d\tau = \text{Fluid memory effect or retardation function}\)
- \((K_{ij}^{\text{hyd}} + K_{ij}^{\text{tower}})\) Combined stiffness matrix, consisting of the hydrostatic restoring matrix and the stiffness of the topside
- \(F_{ij}^{\text{Mooring}}\) = Station keeping restoring force vector.
- \(C_{ij}^{\text{tower}}\) = Damping matrix of the beam element representing the tower.

The right-hand side describes the excitation force by:
- \(F_{i}^{\text{Exc}_\text{inertial}} + F_{i}^{\text{Exc}_\text{viscous drag}}\) = Wave excitation force vectors consisting of the inertial Froude-Krylov and diffraction force and the relative viscous drag part of Morison’s equation.

The equation of motion is solved at the interface point between the floater and the tower in the initial position of the structure by applying the ODE45 routine in MATLAB (2011). Individual transformation matrices are applied in order to transform all forces from their respective point of attack to the interface point. The numerical model is further described in Wehmeyer et al. (2014), where the response model is validated by a decay test and the applied drag coefficients, which agree well with commonly used values for cylindrical and angular structures, are reported. For the sake of orientation, the natural frequencies of the structure applied in the current work are given in model scale in Table 1.

It needs to be noted that the original TLP design was made for 100 meters water depth, which was realized in the physical model test by a deep section located in the middle of the test basin. In order to avoid wave refraction, the deep section was covered by a plate with holes leaving just enough space for the tendons to move freely. The resulting prototype water depth was 56 meters, which resulted in nonlinear storm event waves. This circumstance enabled the current investigation of the nonlinear wave impact on the FOWT TLP.

Irregular Sea State

The second-order transfer functions for the generation of the nonlinear irregular waves are based on the complete theory by Schäffer (1996). The derivation and validation of the velocity potentials and respective expressions for the wave kinematics are shown in the Appendix. Hu and Zhao (1993) presented an empirical limit up to which the second-order wave model is valid, as given by Eq. 2:

\[ \frac{\sigma_n}{\lambda_P} < 0.02 \] (2)

where \(\sigma_n\) is the standard deviation of the surface elevation and \(\lambda_P\) the wavelength of the peak period wave. The numerical realization of the numerical linear and nonlinear sea states uses \(H_{\text{so}}\) and \(T_P\) values according to the mean of the measured sea state parameters (Wehmeyer et al., 2013), which resulted in \(\sigma_n/\lambda_P = 0.011\).

<table>
<thead>
<tr>
<th>Surge [Hz]</th>
<th>Heave [Hz]</th>
<th>Pitch [Hz]</th>
</tr>
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<tbody>
<tr>
<td>0.37</td>
<td>2.49</td>
<td>1.92</td>
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</table>

Table 1. Global natural frequencies of the investigated structure for a water depth of around 100 meters

Fig. 2 Normalized wave crest distribution from observation, first- and second-order irregular sea state realizations, and respective distribution models

An extensive description of the physical model tests can be found in Wehmeyer et al. (2013). As stated before, the numerical water depth is taken according to the physical wave generation water depth. Three JONSWAP spectra with constant \(H_{\text{so}}\) and \(T_P\) but with varying \(\gamma\) values of 1.25, 2.50, and 4.48 are used to simulate the numerical sea states in accordance with the physical wave generation. The wave elevation and kinematics for linear and nonlinear realizations are computed by using a random phase and randomized frequency approach. 26 frequencies are used and 20 time series of 60 seconds each are computed and composited into a sea state, thus representing three hours in a model scale of 1:80. The procedure is repeated three times for each of the three \(\gamma\) values, thus resulting in nine linear sea states and nine nonlinear sea states. This may not be entirely statistically stable; however, the total number of sea states is limited by time restrictions. In Fig. 2, a wave crest distribution from the tank test is compared to one of the first-order wave realizations and to one of the second-order wave realizations and is compared against their respective distribution models. By visual inspection, the second-order distribution compares better to observations and the distribution given by Forristall (2000) than the linear distribution. The wave particle velocities are stretched to the instantaneous water surface by Wheeler stretching. A sensitivity analysis of the effect of different stretching approaches is not carried out in the current work.

Embedded Stream-Function Wave

It is common practice in the offshore wind industry to generate storm time series with a shorter duration than three hours. In order to ensure that the design wave and its nonlinearity are covered, the highest wave of the random sea state is replaced by the nonlinear design wave (see Fig. 3). The single design wave is modeled by a Stream-function wave (Fenton, 1988) and subsequently embedded into a reduced irregular time series, typically around 300 seconds.

The transition between the random sea and the Stream-function wave is defined by a sine-cosine transition. In the current work, 1.5 times the wavelength of the design wave is embedded in the irregular wave train in order to ensure a representation of the wave troughs. 0.25 times the wavelength of the total embedded wave is chosen as the overlap, which defines the transition region. A smooth transition between the background sea state and the Stream-function wave is ensured by a multiplication with \(\sin(0)\) to \(\sin(\pi)\) and \(\cos(0)\) to \(\cos(\pi)\) of the respective elevation and kinematics in
which resulted in three distinct maximum wave heights. In addition, wave (SF) shown as a black line. The cyan and red lines mark the respective maximum wave embedded. Each nonlinear irregular numerical sea states are generated for each chosen peak enhancement factor ƒ. The final wave time series including the embedded wave transition between the original second-order background sea state and the embedded Stream-function wave. The resulting wave time series is shown by a green dashed line.

The time domain shows the probability of exceedance (POE) of peak tendon load ratios within one period, i.e., the maximum numerically predicted load within one wave period is normalized by the overall maximum observed load. The physical wave realizations resulted in a marginal scatter of 10−4.

which are used to calculate a running average in order to find a convergence value (see Fig. 4).

RESULTS

The numerically obtained bow tendon loads serve as an indicator of the numerical model performance. For each of the three ƒ values, which have been used according to the physical sea state generation, the bow tendon load results are shown in Fig. 5 by a normalized time domain representation in the upper row and by a frequency domain representation in the lower row. The observed tendon load signal is given in black. The numerical tendon load signal due to a first-order incident sea state realization is given in magenta. The numerical tendon load signal due to a second-order incident sea state realization is given in blue. No Stream-function wave is embedded at that stage, which means that the highest wave in each numerical sea state is random, as is the highest bow tendon load.

The time domain shows the probability of exceedance (POE) of peak tendon load ratios within one period, i.e., the maximum numerically predicted load within one wave period is normalized by the overall maximum observed load. The physical wave realizations resulted in a marginal scatter of 10−4.

Three irregular wave runs have been conducted in the physical model test, one for each chosen peak enhancement factor (γ), which resulted in three distinct maximum wave heights. In addition to the three-hour irregular sea states described above, twenty numerical sea states are generated for each γ value, but now with the respective maximum wave embedded. Each nonlinear irregular sea state with the embedded wave has a length of 270 seconds (in prototype scale). These three times twenty incident sea states result in three times twenty maximum values of the bow tendon load.

<table>
<thead>
<tr>
<th>γ</th>
<th>1.25</th>
<th>2.5</th>
<th>4.48</th>
</tr>
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<tr>
<td>H_{max}/H_{m0}</td>
<td>1.86</td>
<td>1.73</td>
<td>1.63</td>
</tr>
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</table>

Table 2 $H_{max}/H_{m0}$ ratios from observations

$$S_{I} = \frac{2\pi H_{m0}}{g \left(\frac{T_{i}}{2}\right)^2}$$

where $T_{i}^2$ is the square of the mean wave period in the spectrum.
Fig. 5 Upper row from left to right: Probability of exceedance of normalized bow tendon loads for $\gamma = 1.25$, $\gamma = 2.50$, and $\gamma = 4.48$, given for observations, first-order incident wave realization, and second-order incident wave realization. Loads are normalized by the maximum observed bow tendon load and scaled to match the observed steepness. Lower row from left to right: Power spectra of bow tendon loads for $\gamma = 1.25$, $\gamma = 2.50$, and $\gamma = 4.48$, given for observations, first-order incident wave realization, and second-order incident wave realization. It can be seen in Fig. 5 that the $\gamma$ value specific maximum measured tendon load decreases with increasing $\gamma$ value. This is in line with the expectation that higher $\gamma$ values cause less steep waves, which result in fewer overturning moments around the still water line. A nonzero value can be observed for a POE of 1, which is due to the pretension. Between 90% and 95% of the tendon load signals are predicted satisfactorily by the numerical model. The remaining 5% to 10% are overpredicted by the numerical model, where the mean ratio of the nine first-order load predictions is found to be 1.32 and the mean ratio of the nine second-order load predictions is found to be 1.34 (see Table 3).

The frequency domain results of the bow tendon load (see the lower row in Fig. 5) show a satisfactory match in the range between 0.5 Hz and 1.25 Hz in model scale, and the different spectral shapes due to the different $\gamma$ values are obvious. Pitch and heave natural frequencies are easily identifiable; however, a distinct surge natural frequency is neither seen in the physical model nor in the numerical model. A significant lack of energy is observed between 1.3 Hz and 2 Hz, i.e., close to the pitch natural frequency, whereas an excess of energy can be observed between 2 Hz and 2.7 Hz, i.e., close to the heave natural frequency.

The maximum bow tendon loads obtained by the incident embedded Stream-function wave show an expected scatter from the

<table>
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<th>$\gamma$</th>
<th>1.25</th>
<th>2.5</th>
<th>4.48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean First Order</td>
<td>1.37</td>
<td>1.30</td>
<td>1.30</td>
</tr>
<tr>
<td>Mean Second Order</td>
<td>1.33</td>
<td>1.32</td>
<td>1.37</td>
</tr>
</tbody>
</table>

Table 3 Mean values of maximum bow tendon load ratios for each $\gamma$ over three runs; total mean value over all $\gamma$ and mean value from Stream-function runs.

twenty $\gamma$ values for specific realizations. The embedded waves match the maximum measured waves and show different crest front steepness values (Myrhaug and Kjeldsen, 1986). Being a direct response to the maximum wave, the obtained load ratios, again normalized by the overall maximum measured load, follow the trend of the crest front steepness values (see Table 4). The total mean value of the Stream-function runs, averaged over all $\gamma$ values, is 1.29 (see Table 3). No steepness scaling is applied for this case. The running average converges to the mean values presented in Table 4 after approximately 18 runs (see Fig. 5); however, a small deviation of the last value can still be observed, which indicates the need for more runs. The load ratios from all three $\gamma$ value runs are used to fit a normal distribution and to determine the area resembling 98% coverage, i.e., $10^{-4}$. The remaining 2% is indicated by the encircled red area (see Fig. 4).

DISCUSSION

A shortcoming of the embedded wave approach is that the measurements were actually made in front of the structure. Therefore, the measured waves do not exactly resemble the ones impacting on the structure, which makes it difficult to define the exact wave that
is responsible for the highest measured impact. The most interesting load ratio value is therefore the one from the lowest γ value, where the ratio between the measured $H_{m0}$ and measured $H_{max}$ is very close to current design requirements (IEC, 2009). In that case, the mean load ratio converges towards 1.37 and is equal to the highest average ratio of the irregular sea state runs. For the remaining γ values, the mean load ratios follow the crest front steepness values, which is expected (see Table 4). The approach outlined by Myrhaug and Kjeldsen (1986) to determine the crest front steepness was originally intended to describe wave crest asymmetries around the horizontal and vertical axes. A Stream-function wave is symmetrical around the vertical axis; however, the reason for applying this approach was the strong asymmetry of a Stream-function wave around the horizontal axis. Hence, the applied steepness description considers the crest height, which is believed to be an important parameter considering a structure response that is sensible to pitch motions resulting from hydrodynamic drag up to the free surface.

A Gaussian distribution is fitted through the maximum bow tendon load ratio scatter originating from the embedded Stream-function waves. The ratio between the 98th percentile, defined by an area of $10^{-4}$ in the probability density function (PDF), and the measured value generally determines the safety level. In this case, a ratio below 1 would be required (see Fig. 4), which highlights the degree of conservatism of the current approach.

The overall mean load ratio values, which result from the first- and second-order incident wave realizations, follow the expected trend, in which the second-order responses are slightly higher (see Table 3). However, unexpected high values for γ values of 1.25 and 4.48 for the first- and second-order incident wave realizations, respectively, are found. This highlights the need for a higher number of runs in order to obtain a higher degree of statistical stability.

The influences of the nonlinear incident wave realizations can be seen only for the highest 5% to 10% of the bow tendon load events. This is expected, as the higher crests of the second-order realizations result in higher overturning moments. The spectral energy contents of the numerical bow tendon loads do not show any distinct differences upon application of the linear and nonlinear incident wave realizations. This is not surprising, as the higher-order wave model mainly influences the highest waves; its influence on the spectral energy is marginal. The measured spectral energy peak around 1.0 Hz (see the middle plot in the lower row in Fig. 5) is assumed to be due to a yaw motion, which cannot be captured by the 2D numerical model. However, most of the observed differences are in the high frequency region, and it is consequently believed that the missing second-order sum frequency inertial forces cause the difference. This is in line with the findings of other authors and underlines the importance of further model improvements, i.e., the inclusion of those higher-order inertial forces.

CONCLUSIONS

Not surprisingly, the higher-order irregular wave generation is well established, and good agreement can be seen between the distributions of the numerical results and measurements. The embedment of a Stream-function wave is a common industry practice. It provides measures to control the highest wave in an otherwise random sea state, which is very beneficial for engineering applications. The embedment into a second-order background sea state is new, but results from the fact that a second-order sea state is more realistic, considering the enhanced higher-order terms for large sea states. To the authors’ knowledge, it is the first time that an embedded Stream-function wave is used as an incident wave for a FOWT. The approach is certainly controversial due to the violation of linear theory. However, the intention is not so much to accurately depict the observed responses; rather, the intention is much more to assess how a somewhat simple numerical hybrid model reacts if exposed to nonlinear wave elevations and kinematics. In the short-term future, the fundamentally new challenge of installing dynamically sensitive FOWTs in limited water depths will require clarification of the applicability of time-efficient solutions to respective design tasks. The current work intends to assess the possibility of this approach and the degree of conservatism that this approach contains. The irregular incident wave runs overpredict the measured maximum loads by 32% on average, where the nonlinear irregular wave runs yield more conservative load ratios. From this, it can be concluded that the linear irregular incident wave is sufficiently conservative even though the nonlinear incident wave model better resembles the surface elevation (and probably the wave kinematics). The embedded wave approach results in slightly lower overall averaged maximum loads, i.e., an overprediction of 29%. Therefore, based on the current status, it is concluded that the embedded wave approach provides a controlled and time-efficient engineering tool for the investigated FOWT. However, optimizations are required in order to reduce the degree of conservatism.

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REFERENCES


APPENDIX

The theoretical background for the computation of a nonlinear time series is given in Schäffer (1994) and later on in Skourup and Sterndorff (2002). However, the final equations are not given and are therefore presented below as they have been implemented.

Surface Elevation

It is assumed that the nonlinear surface elevation can be expressed as a summation of the first- and second-order components. Hence, the surface elevation is expressed as:

$$\eta^{(1+2)} = \eta^{(1)} + \eta^{(2+) + \eta^{(2-)}}$$

The superscripts indicate the well-known first-order solution (1) and the second-order solution (2+) and (2−). The two latter expressions thereby indicate that second-order solution is a superposition of a difference frequency term (i.e., sub-harmonics in the following superscript “−”) and a sum frequency term (i.e., super-harmonics in the following superscript “+”).

The first-order wave elevation is given in complex notation by:

$$\eta^{(1)}(x, t) = \sum_{n=1}^{N} A_n e^{i(\omega_n t - k_n x)}$$

with

$$A_n = a_n e^{i(\delta_n)}$$

$$a_n = \sqrt{2S(\omega_n)} \Delta \omega$$

where

$$A_n$$ = complex amplitude of the nth wave

$$a_n$$ = frequency-respective linear wave amplitude, determined from a spectrum

$$S(\omega_n)$$ = spectral energy density

$$\delta_n$$ = random phase angle

$$\omega_n$$ = wave frequency

$$k_n$$ = wave number

$$i = \text{imaginary unit}$$

c.c. = complex conjugate

The second-order wave elevation for the sub- and super-harmonics is given in complex notation by:

$$\eta^{(2\pm)}(x, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5(G_{nm}^\pm A_n A_m e^{i(\delta_n \mp \delta_m)} + \text{c.c.})$$

with

$$A_n^\pm = \text{complex amplitude of the } m^{\text{th}} \text{ wave}$$

$$(\theta_n \mp \theta_m) = (\omega_n \mp \omega_m) t - (k_n \mp k_m) x$$

and the following coefficients:

$$G_{nm}^\pm = \frac{\gamma_{nm} \delta_{nm}}{2} \begin{cases} (\omega_n \mp \omega_m) H_{nm}^\pm - L_{nm}^\pm & \text{for } n \neq m \\
(\omega_n \mp \omega_m) H_{nm}^\pm - (\omega_n \mp \omega_m) L_{nm}^\pm & \text{for } n = m \end{cases}$$

$$H_{nm}^\pm = (\omega_n \mp \omega_m) \left( \mp \omega_n \omega_m - \frac{g^2 k_n k_m}{\omega_n \omega_m} \right) + \omega_n^2 \mp \omega_m^2 - \frac{g^2}{2} \left( \frac{k_n^2}{\omega_n^2} \mp \frac{k_m^2}{\omega_m^2} \right)$$

$$L_{nm}^\pm = g(k_n \mp k_m \tan(k_n \mp k_m) h - (\omega_n \mp \omega_m)^2$$

$$D_{nm}^\pm = \frac{1}{2} \left( \frac{g^2 k_n k_m}{\omega_n \omega_m} \mp \omega_n \omega_m - (\omega_n^2 \mp \omega_m^2) \right)$$

Kinematics

The kinematics and pressures are well-known derivatives of the velocity potential. Following the approach for the surface elevation, it is assumed that, based on a spectral wave description, all relevant wave parameters can be expressed as a summation of the first- and second-order terms of a velocity potential $\varphi$ and its derivatives according to:

$$\varphi^{(1+2)} = \varphi^{(1)} + \varphi^{(2+)} + \varphi^{(2−)}$$

The first-order velocity potential is given in complex notation by:

$$\varphi^{(1)}(x, z, t) = \sum_{n=1}^{N} \sqrt{2} g A_n \cos \left( k_n (z - h) + \delta_n \right) = \sum_{n=1}^{N} \sqrt{2} g A_n \cosh \left( k_n (z - h) \right) e^{i(\delta_n - k_n x)} + \text{c.c.}$$

Similar to the surface elevation, the second-order velocity potential needs to have two contributions. These are obtained by a summation of two first-order components and the difference of the two first-order components as follows:

$$\varphi^{(2\pm)}(x, z, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5 \left( i \gamma_{nm} F_{nm} A_n^\pm \cosh \left( k_n \mp k_m \right) (z + h) \cosh \left( k_n \mp k_m \right) h \right)$$

with

$$\delta_{nm} = \begin{cases} \frac{1}{2}, & n = m \\ 1, & n \neq m \end{cases}$$

The coefficient $F$ is given by:

$$F_{nm} = H_{nm}^\mp / D_{nm}^\pm$$
For the sub-harmonics, it is seen that $D_{n,m}^+$ tends to zero and hence would cause $F_{n,m}^+$ to tend to infinity. It is assumed to be appropriate to step back to the Laplace equation and boundary conditions for this particular case, to solve these under the assumption that there is no oscillation, and hence to obtain a solution that involves a current and a change in sea level. Both contributions are included in the calculations according to Stokes’ second-order theory.

Particle velocities and accelerations to the first order can be obtained as derivatives of the velocity potential as follows:

\[
\begin{align*}
\frac{\partial \varphi^{(1)}}{\partial x} &= u^{(1)} \\
&= \sum_{n=1}^{N} 0.5 \left( g k_n A_n \frac{\cosh(k_n(z + h))}{k_n h} e^{i(k_n - k_m) + c.c.} \right) \\
\frac{\partial \varphi^{(1)}}{\partial z} &= w^{(1)} \\
&= \sum_{n=1}^{N} 0.5 \left( g k_n A_n \frac{\sinh(k_n(z + h))}{k_n h} e^{i(k_n - k_m) + c.c.} \right) \\
\frac{\partial u^{(1)}}{\partial t} &= \dot{u}^{(1)} \\
&= \sum_{n=1}^{N} 0.5 \left( i g k_n A_n \frac{\cosh(k_n(z + h))}{k_n h} e^{i(k_n - k_m) + c.c.} \right) \\
\frac{\partial w^{(1)}}{\partial t} &= \dot{w}^{(1)} \\
&= \sum_{n=1}^{N} 0.5 \left( -g k_n A_n \frac{\sinh(k_n(z + h))}{k_n h} e^{i(k_n - k_m) + c.c.} \right)
\end{align*}
\]

Wave-induced excess pressure to the first order is given by:

\[
\begin{align*}
\frac{p^{(1)}}{\rho} &= -\frac{\partial \varphi^{(1)}}{\partial t} \\
\end{align*}
\]

with

\[
\begin{align*}
\frac{\partial \varphi^{(1)}}{\partial t} &= \sum_{n=1}^{N} 0.5 \left( -g A_n \frac{\cosh(k_n(z + h))}{k_n h} e^{i(k_n - k_m) + c.c.} \right)
\end{align*}
\]

Particle velocities and accelerations to the second order can be derived accordingly as:

\[
\begin{align*}
\frac{\partial \varphi^{(2+)}}{\partial x} &= u^{(2+)} = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5 \left( \delta_{n,m} F_{n,m}^{+} A_n A_m^{+} (k_n + k_m) \right) \\
&\quad \times \cosh((k_n + k_m)(z + h)) e^{i\theta_n + \theta_m} + c.c. \\
\frac{\partial \varphi^{(2+)}}{\partial z} &= w^{(2+)} = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5 \left( \delta_{n,m} F_{n,m}^{+} A_n A_m^{+} (k_n + k_m) \right) \\
&\quad \times \cosh((k_n + k_m)(z + h)) e^{i\theta_n + \theta_m} + c.c. \\
\frac{\partial \varphi^{(2-)}}{\partial x} &= u^{(2-)} = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5 \left( \delta_{n,m} F_{n,m}^{-} A_n A_m^{-} (k_n - k_m) \right) \\
&\quad \times \cosh((k_n - k_m)(z + h)) e^{i\theta_n - \theta_m} + c.c. \\
\frac{\partial \varphi^{(2-)}}{\partial z} &= w^{(2-)} = \sum_{n=1}^{N} \sum_{m=1}^{N} 0.5 \left( \delta_{n,m} F_{n,m}^{-} A_n A_m^{-} (k_n - k_m) \right) \\
&\quad \times \cosh((k_n - k_m)(z + h)) e^{i\theta_n - \theta_m} + c.c. \\
\end{align*}
\]