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Mathematical Modeling of Shock Loading of a Solid Ice Cover

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This paper deals with the linear 2-dimensional task concerning the effect of an impulsive load on a viscoelastic ice plate on the elastic basis, i.e., water. Analysis is made of the following factors: the effect of ice-plate thickness, relaxation time, variable depth of a basin, remoteness and steepness of the coastal bottom-surface slope, and choice of a function type of an instantaneous impulsive load on the amplitude of the plate deflection.

INTRODUCTION

The conduct of ice under heavy loads originating in explosions, i.e., impact impulses, is not sufficiently investigated. Nevertheless, this problem is of great economic value, as exploding an ice cover is used when constructing runways, smoothing out hummocks on the ice surface, breaking down ice bonds and heavy hummocked ice when building shipping canals by ice-breakers, expanding passageways between 2 ice caps, and in their slivering, in squeezing ships by heavy ice, etc. For estimating this technology's efficiency and optimization, both experimental and appropriate theoretical investigations are necessary. The purpose of this paper is to consider the 2-dimensional task concerning the conduct of an ice cover after it is affected by an instantaneous impulsive load. Earlier, Fox (1993) and Kerr (1976) considered a similar problem, but without accounting for a coastal bottom-surface slope. The book *Moving Loads on Ice Plates* by Squire et al. (1996) includes an extensive bibliography on a similar theme.

MATHEMATICAL STATEMENT

Here we consider a floating and infinite ice plate which at first is at rest in a state of equilibrium and is then actuated in an instant $t=0$ by a shock loading. The coordinates are positioned as follows: The coordinate basic origin is combined with the impulse application point, the axis Ox coincides with the unperturbed ice-water interface, and the axis Oz is directed vertically upward. It is assumed that water is an ideal, incompressible fluid of density ρ_2, and fluid motion is potential. The ice field is simulated by an initially unstressed, viscoelastic, homogeneous, isotropic plate. According to Kheisin (1967), the Kelvin-Voigt law of deformation of a delayed-elastic linear medium (Freudenthal and Heiringer, 1962) is used for ice. In this case, the differential equation of small vibrations of the floating plate will be as follows:

$$\frac{Gh^3}{3}\left(1+\tau_\phi\frac{\partial}{\partial t}\right)\frac{\partial^4 w}{\partial x^4}+\rho_1 h\frac{\partial^2 w}{\partial t^2}+\rho_2 g w+\rho_2\frac{\partial \Phi}{\partial t}\bigg|_{z=0} = -Y(x)\delta(t-0) \qquad (1)$$

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where G is the shear elastic modulus of ice; $G=0.5E/(1+\nu)$, E is the elastic modulus of ice in tension and compression, and ν is Poisson's constant; $h(x)$ is the ice thickness, $\rho_1(x)$ the ice density, and τ_ϕ the strain relaxation time for ice, or the "delay time" (Kheisin, 1967; Freudenthal and Heiringer, 1962); $w(x,t)$ is the displacement of the fluid surface, or the vertical displacement of ice; $\Phi(x,z,t)$ is the fluid velocity's potential function, satisfying the Laplace equation $\Delta\Phi=0$; $Y(x)$ is an instantaneous impulsive load function; and $\delta(t-0)$ is the Dirac delta function. Below, we assume that ρ_1 and h are constants. The reduced values of the shear modulus G and the ice density ρ_1 found by integration over the plate thickness should be taken as the calculated values.

The Dirac delta function $Y_0\delta(x)$, or the centered function of normal pressure distribution $Y_0(a/\pi)\exp(-ax^2)$, is considered the instantaneous impulsive load function $Y(x)$.

Fig. 1 shows the bivariative centered function of normal pressure distribution $Y(x)=5\cdot 10^4(a/\pi)\exp(-ax^2)$ for 3 different values of parameter a. It can be seen that, with the increase of parameter a, the pressure areas' section of an impulsive load decreases while the load itself gains in amplitude. When $a\to\infty$, the centered function of an instantaneous impulsive load's normal distribution tends to delta function $\delta(x)$.

The initial conditions for w will be homogeneous:

$$w(0)=\dot{w}(0)=0 \qquad (2)$$

The linearized kinematic condition at the ice-water interface has the following form:

$$\frac{\partial \Phi}{\partial z}\bigg|_{z=0}=\frac{\partial w}{\partial t} \qquad (3)$$

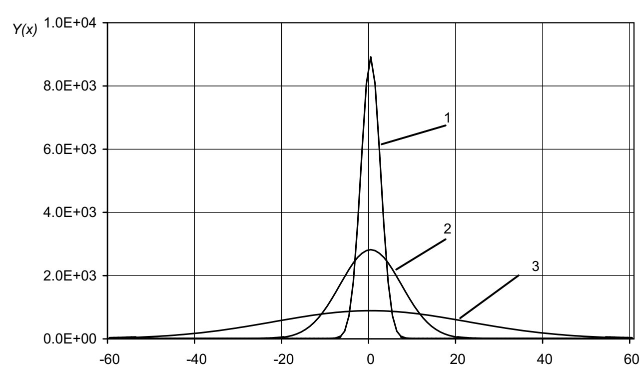

Fig. 1 Centered function of normal pressure distribution at $Y_0=5\cdot 10^4$ for different a: (1) $a=0.1$; (2) $a=0.01$; and (3) $a=0.001$

The boundary condition at the basin bottom for the function $\Phi(x, z, t)$ is written as follows:

$$\frac{\partial \Phi}{\partial \vec{n}} = 0 \quad \text{for} \quad z = -H, \tag{4}$$

Here, $H = H_1 - b$; where H_1 is the basin depth, $b = \rho_1 h/\rho_2$ is the ice immersion depth in static equilibrium. It is assumed that the slope of the bottom surface along the direction of axis Ox equals α. And the value α is so small that the value H can be considered constant when making calculations in each concrete point x.

ANALYTICAL SOLUTION

Following Kozin and Pogorelova (2004), we assume that the task can be solved by using the Fourier transformations on coordinate x for functions $w(x, t)$ and $\Phi(x, z, t)$. The transforms of functions $w(x, t)$ and $\Phi(x, z, t)$ on variable x are considered:

$$w_F(\gamma, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i(\gamma x)) w(x, t)\, dx,$$

$$\Phi_F(\gamma, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-i(\gamma x)) \Phi(x, z, t)\, dx$$

After application of a Fourier transformation to Eq. 1 and usage of kinematics and boundary conditions (Eqs. 3 and 4), we can receive the inhomogeneous differential 2nd-degree equation with stationary values of coefficients for a transform w_F:

$$\ddot{w}_F m(\gamma) + \dot{w}_F k(\gamma) + w_F c(\gamma) = -Y_F \delta(t - 0), \tag{5}$$

where:

$$k(\gamma) = \tau_\phi \frac{G h^3 \gamma^4}{3}, \quad m(\gamma) = \rho_1 h + \frac{\rho_2 (1 + \alpha^2)}{\gamma(\text{th}(\gamma H) + \alpha^2 \text{cth}(\gamma H))},$$

$$c(\gamma) = \rho_2 g + \frac{G h^3 \gamma^4}{3}, \quad Y_F = \begin{cases} \dfrac{Y_0}{\sqrt{2\pi}} \\ \dfrac{Y_0}{\sqrt{2\pi}} \exp\left(-\dfrac{\gamma^2}{4a}\right) \end{cases}$$

Here, the function type Y_F depends on the choice of function of an instantaneous impulsive load $Y(x)$, and the values α and H are considered constant and independent of x.

Applying the Laplace transformation to the solution of Eq. 5 under homogeneous initial conditions, we obtain:

$$w_F = \begin{cases} \dfrac{-Y_F}{\sqrt{cm - k^2/4}} \exp\left(-\dfrac{kt}{2m}\right) \sin\left(\dfrac{t}{m}\sqrt{cm - \dfrac{k^2}{4}}\right) & \text{for } cm - k^2/4 > 0 \\ \dfrac{-Y_F}{\sqrt{k^2/4 - cm}} \exp\left(-\dfrac{kt}{2m}\right) \text{sh}\left(\dfrac{t}{m}\sqrt{\dfrac{k^2}{4} - cm}\right) & \text{for } k^2/4 - cm > 0 \\ \dfrac{-Y_F}{m} t \exp\left(-\dfrac{kt}{2m}\right) & \text{for } cm - k^2/4 = 0 \end{cases} \tag{6}$$

By analogy with Kheisin (1967) and Kozin and Pogorelova (2004), the looked-for function w is found using the inverse Fourier transformation:

$$w(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty w_F \gamma J_0(\gamma x)\, d\gamma \tag{7}$$

Here, the value of w_F is calculated by Eq. 6; $J_0(\gamma x)$ is the 1st-order Bessel function; the values of c, m, Y_F and k are calculated by Eq. 5.

NUMERICAL ANALYSIS OF OUTCOMES

The results of calculations by Eq. 7 were analyzed depending on the distance to the impulse application point and on the time passed from the moment of the impulse application, for the following parameters of the ice plate and water:

$$\rho_2 = 1000 \text{ kg/m}^3; \quad \rho_1 = 900 \text{ kg/m}^3; \quad E = 5 \cdot 10^9 \text{ N/m}^2;$$

$$\nu = \frac{1}{3}; \quad h = 0.5 \div 2.0 \text{ m}; \quad \tau_\phi = 0.69 \text{ s}.$$

The data of the value are close to actual ice parameters. The effect of the strain relaxation time on the value of the plate's vibration amplitude has been analyzed by Kozin and Pogorelova (2004). It is shown that the viscoelastic model of ice gives a good fit to the experimental data. The magnitude of relaxation time τ_ϕ was selected according to conclusions reached by Squire et al. (1996) and Takizava (1985). The absolute value of an impact impulse was assumed to be equal to $Y_0 = 5 \cdot 10^4$ kg/s.

Fig. 2 shows the effect of parameter a on the vertical displacement of the ice plate w at the point of pulse application $x = 0$ depending on time t for $H = 5$ m, $h = 0.5$ m. Here, curves 1, 2 and 3 are plotted for $a = 0.001, 0.01, 0.1$, respectively, for the representation of an impulse load $Y(x)$ as the central function of a normal distribution; curve 4 corresponds to parameter values $a \geq 1$ and coincides with a solution for a delta function.

From Fig. 2 it follows that, with the exception of very small parameter values a ($a \leq 0.001$) at $\tau_\phi = 0.69$ s, the ice plate's maximum deflection is at the point of impulse application ($x = 0$) in an instant $t = 1$ s (provided that $H = \text{const}$).

An analysis of Fig. 2 shows that the deflection of the ice plate when acted on by an impulse load given on a normal pressure distribution function with the increase of a tends to a deflection caused by a delta function. Thus, with the ice thickness $h \geq 0.5$ m, the radius of the impact load area acting on the ice plate is less than 1 m (approximately corresponding to $a = 1$), and it is possible in linear statement to select function $\delta(x)$ as the function of an impact load on ice.

Fig. 3 shows the effect of ice thickness h and basin depth H on the plate's deflection magnitude w at the point $x = 0$ depending on time t for $a = 10$. Here, curves 1, 2 and 3 are plotted for $H = 5$ m, 10 m, 50 m, respectively, at $h = 0.5$ m; curves 4 and 5 correspond to ice-plate thickness $h = 1$ m, 2 m for basin depth $H = 5$ m.

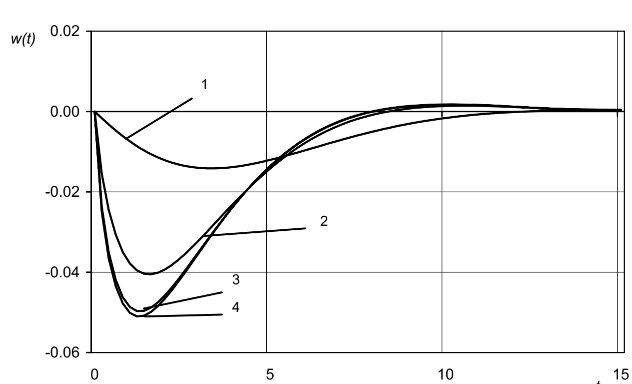

Fig. 2 Ice deflection w as function of t for various aspects of function of an impulsive load: 1, 2 and 3 = centered function of a normal pressure distribution for $a = 0.001$, 0.01, 0.1; 4 = delta function

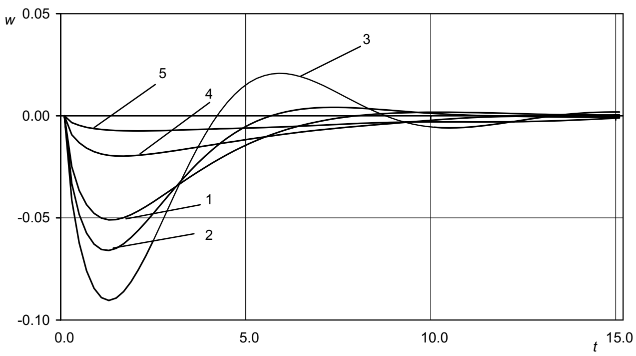

Fig. 3 Ice deflection w as function of t for different water depth H and ice-plate thicknesses h: (1) $H = 5$ m, $h = 0.5$ m; (2) $H = 10$ m, $h = 0.5$ m; (3) $H = 50$ m, $h = 0.5$ m; (4) $H = 5$ m, $h = 1$ m; (5) $H = 5$ m, $h = 2$ m

From Fig. 3 it follows that, in the case of final basin depth H, the increase of basin depth H and the decrease of ice-plate thickness h result in the plate's deflection increase.

The results shown in Figs. 2 and 3 were obtained on condition that the basin depth H remains constant for different distances x, and the slope of the bottom surface is $\alpha = 0$.

For the analysis of the action of an instantaneous impulsive load on ice cover near shore, one can assume that the basin depth's function H varies under the following:

$$H = 0.5H_0(1 + \text{th}(\mu(L - x))) \tag{8}$$

Here, H_0 is basin depth on infinite moving from the shore; L is the distance from an impulse application point $x = 0$ up to an inflection point of the coastal bottom surface slope in the direction x; μ is a parameter adequate for the steepness of the bottom surface slope when approaching a coast. The present Eq. 8 of depth H allows linear statement to simulate the coastal bottom surface slope while remaining within the framework of an infinite plate.

Fig. 4 shows the function of basin depth H, calculated using Eq. 8 at $H_0 = 5$ m, $L = 12.5$ m depending on distance x. Here curves 1∼3 correspond to parameter μ, which is equal to $10^{-0.2}$, $10^{-0.5}$, $10^{-0.8}$. It can be seen that the more magnitude μ has, the steeper the bottom surface coast slope is.

If using Eq. 7 for calculating an ice-plate deflection depending on x and calculating the basin depth in each concrete point x under Eq. 8, interesting results can be achieved about the effect of a slope angle of the bottom surface on the ice plate's vibration amplitude.

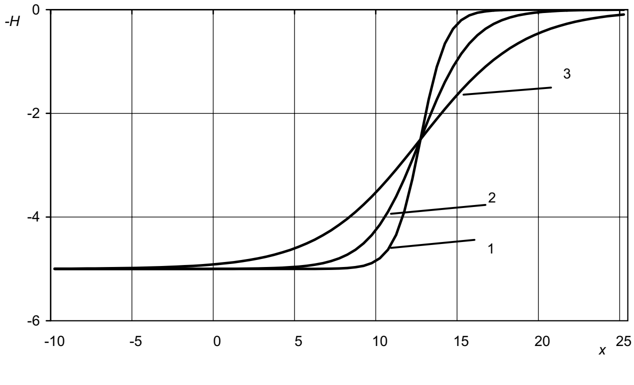

Fig. 4 Function of depth of basin H at $H_0 = 5$ m, $L = 12.5$ m for different μ: (1) $\mu = 10^{-0.2}$; (2) $\mu = 10^{-0.5}$; (3) $\mu = 10^{-0.8}$

Figs. 5a∼c show the results of calculations by Eq. 7 in view of Eq. 8. Here, curve 1 corresponds to ice-plate deflection at $H = \text{const} = 5$ m, $t = 1$ s, $a = 10$ and $h = 0.5$ m. Curves 2∼5 correspond to time t, which is equal to 1 s, 3 s, 5.5 s and 10 s for $a = 10$, $h = 0.5$ m and to basin depth function calculated under $H = 2.5(1 + \text{th}(10^{-0.5}(L - x)))$.

Fig. 5a corresponds to a case where the distance from an impulse application point to an inflection point of the coastal bottom surface slope is $L = 10$ m; Fig. 5b, to $L = 5$ m; Fig. 5c, $L = 0$ m.

From Figs. 5a∼c it follows that the presence of a coast sloping near the application point of an instantaneous impulsive load results in an ice plate's deflection increase. The greatest plate deflection can be seen not in area located above the inflection point of the coast's slope line and relevant to the maximum slope angle of the bottom surface, but in the area somewhat displaced from the inflection point to shallow water. And the less the distance L, i.e., the nearer the coastal slope's inflection point is to

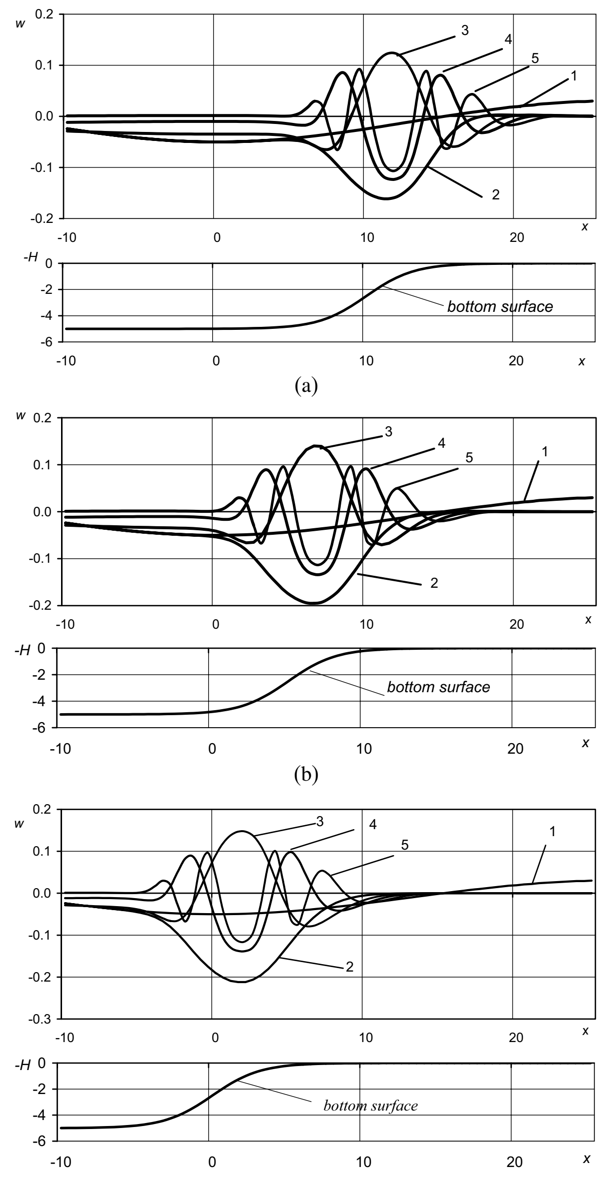

Fig. 5 Ice deflection w as a function of x for various distances L from an impulsive point up to an inflection point of coast bottom's surface slope line: (a) $L = 10$ m; (b) $L = 5$ m; (c) $L = 0$ m

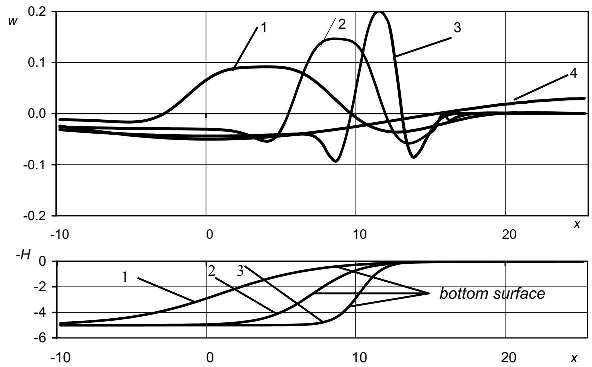

Fig. 6 Effect of parameter μ on a deflection of ice plate w: (1) $\mu = 10^{-0.8}$, $t = 5.7$ s, $L = 0.81$ m; (2) $\mu = 10^{-0.5}$, $t = 3.4$ s, $L = 6.93$ m; (3) $\mu = 10^{-0.2}$, $t = 2.25$ s, $L = 10$ m; (4) $\mu = 0$, $t = 1$ s

the impact impulse, the greater the ice plate's deflection amplitude. In due course the flexural-gravity wave caused by an impact impulse is transformed into waves going in the direction of the shore, reflected from the shore and damping on amplitude.

Fig. 6 shows the effect of parameter μ, which is responsible for the steepness of the bottom-surface coast slope, on the ice plate's deflection magnitude w. Outcomes are obtained in Eq. 7 in view of Eq. 8 for $H_0 = 5$ m, $h = 0.5$ m and $a = 10$. Here, curve 4 corresponds to ice-plate deflection at $H = \text{const} = 5$ m, $t = 1$ s, $a = 10$ and $h = 0.5$ m. Curves 1~3 are plots of the dependence $w(x)$ for parameter $\mu = 10^{-0.8}$, $10^{-0.5}$, $10^{-0.2}$, respectively.

Analyzing Fig. 6, one can come to the conclusion that growth of parameter μ (increase in the steepness of the bottom surface's coastal slope) results in the amplitude increase of an ice plate's deflection.

CONCLUSIONS

This paper deals with the effect of the ice-plate thickness, basin depth, and proximity and steepness of the coastal bottom surface slope on the amplitude of the flexural-gravity wave which is caused by an impact impulsive load action on an infinite ice plate. The task was solved in linear statement. It turned out that the action of an instantaneous impulsive load on ice as the centered function of a normal pressure distribution $Y_0(a/\pi)\exp(-ax^2)$ for $a \geq 1$ is identical to delta function action $Y_0\delta(x)$. Increase in basin depth results in amplitude growth of the ice-plate deflection. On the contrary, increase in ice-plate thickness reduces the plate's deflection amplitude. For a flat ground ($H = \text{const}$), the maximal plate deflection is seen in the impulse application point. For a rough bottom $H = 0.5H_0(1 + \text{th}(\mu(L - x)))$, the plate deflection is jointly affected by the distance to the inflection point of the coastal slope, by the coastal slope's steepness, and by the basin depth. It is advisable to compare the obtained theoretical results with the experimental data.

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