

The Statistical Distribution of a Nonlinear Ocean Surface

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This paper first discusses probability density functions (pdfs) which could be applied to sea surface elevation, both observed in the sea and simulated numerically. The pdfs are grouped into 2 categories: One is derived from probability theory (e.g., the Longuet-Higgins/Edgeworth expansion, the Pearson system and the Maximum Entropy estimate); the other is derived directly from the ocean wave equations (e.g., Stokes wave theory, NLS equation). In order to assess the ability of the distributions to fit the data, we have analyzed numerical simulations of the surface of ocean waves using higher-order NLS-type equations (Dysthe et al., 1979; Trulsen et al., 1996, 2000). For all simulations, the Pearson system shows reasonable agreement with the kernel distribution both at the tail and around the peak. For small wave steepness, the theoretical distributions first derived by Tayfun (1980) and Huang and Long (1983) fit the simulated data well, while these distributions fit the kernel density function less well as the steepness increases. In general, most of the pdfs are in reasonable agreement with the kernel distribution up to 3-4 standard deviations. Measured data behave similarly to simulated data.

INTRODUCTION

It is common practice to model the ocean surface $\eta(\mathbf{x}, t)$ using linear wave theory, for which η is Gaussian. This ignores the well-known asymmetry of real waves, where wave crests tend to be higher than the depth of their neighboring troughs. Thus, a histogram of surface elevations is not symmetric around the mean value, and a positive skewness increases with the steepness of the sea severity for a given water depth. In recent years, higher-order equations of the nonlinear Schrödinger type for ocean waves have been developed. Dysthe (1979) first extended the NLS Equation to 4th order in infinitely deep water. This was followed by extensions to finitely deep water (Brinch-Nielsen and Jonssen, 1986), broader bandwidth (Trulsen and Dysthe, 1996), and exact linear dispersion (Trulsen et al., 2000). A detailed discussion of this and other nonlinear equations of waves can be found in Trulsen (2003). Efficient numerical implementations, starting with the algorithms developed by Lo and Mei (1985), have made it possible to make numerical simulation of regions of the order of 100×100 wavelengths and time spans covering the validity of the equations (~ 150 peak wave periods).

Statistical analysis of ocean waves as well as various remote sensing methods (e.g., satellite altimetry) require accurate statistical distributions of the surface elevation. Since the data material will always be incomplete, a definite answer is not possible.

It is well known that the probability density function for the surface elevation is close to the Gaussian distribution, and several families of distributions have been applied in the past. The families belong to 2 categories: One is derived from probability

theory, examples of which are the Longuet-Higgins and Edgeworth expansions, the Pearson system, and Maximum Entropy estimates; the second, from nonlinear wave theory, e.g., Stokes theory (Tayfun, 1980), and distributions derived from the NLS Equations (Socquet-Juglard et al., 2004).

In this study, we first give a short review and discuss the feasibility of the distributions above. The distributions are then fitted to simulated surface elevations in deep water. The simulations utilize the Modified Non-Linear Schrödinger (MNLS) equation and are able to produce the time development of sections of the size 128×128 typical wavelengths and over 150 wave periods for 2-dimensional wave fields, or domains of the size 1024 typical wavelengths and over 150 wave periods for 1-dimensional spatial wave series (Trulsen and Dysthe, 1996; Trulsen et al., 2000). The simulated surfaces are reconstructed to 2nd and 3rd order. We use a kernel distribution of the simulated data as the empirical distribution and compare this with the other distributions that are fitted based on 4 sampled moments. It is concluded that the Pearson system shows quite good agreement with the kernel distributions both at the tails and the main peak for most of the pdfs. In general, most of the distributions are in reasonable agreement with the kernel distribution up to 3–4 standard deviations. Measured data behave similar to simulated data.

CLASSES OF PROBABILITY DISTRIBUTIONS

Let $p(x)$ be a probability distribution with mean μ , and centralized moments:

$$\mu_n^c = \int_{-\infty}^{\infty} (x - \mu)^n p(x) dx, \quad n = 2, 3, \dots \quad (1)$$

The *standard deviation* (σ), *skewness* (γ) and *kurtosis* (κ) are defined:

$$\sigma = \sqrt{\mu_2^c}, \quad \gamma = \frac{\mu_3^c}{\sigma^3}, \quad \kappa = \frac{\mu_4^c}{\sigma^4}. \quad (2)$$

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KEY WORDS: Modified non-linear Schrödinger (MNLS), maximum entropy, Pearson system, Gaussian kernel estimator, Edgeworth expansion.