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A Simple Formula for Nonlinear Wave-Wave Interaction

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ABSTRACT

A simple expression is introduced as an approximation for the rate of change of the spectral energy density of surface gravity waves due to nonlinear wave-wave interaction. It has the form of a second-order nonlinear diffusion operator, and conserves wave energy, momentum and wave action. It is independent of the details of the dispersion relation, so it can possibly be used for both deep and shallow water, although its application to shallow water is not explicitly considered. The directional dependence of the formula is essential in permitting the wave momentum to be conserved, in addition to the wave energy and action. The formula may be useful in discussing the qualitative behavior of wave spectrum evolution without making elaborate calculations. It is consistent with the observed and modelled result that nonlinear effects tend to cause the wave energy to be transferred to lower wave frequencies. However, when applied to a JONSWAP wave spectrum it behaves rather diffusively, tending to directly reduce the amplitude of the spectral peak. In the absence of other wave energy source terms, the formula leads to various time-independent wave spectra, whose dependence on scalar wave number is linked to the angular wave energy distribution. In general the directional spreading of the spectrum tends to increase as the scalar wavenumber increases. The limiting directionally-isotropic spectrum has the Kitaigorodskii equilibrium-range behavior, where the wave energy (variance) spectrum is proportional to the inverse fourth power of the frequency.

INTRODUCTION

Spectral wave forecasting models have been used for many years to predict directional wave spectra worldwide, in the open ocean and in coastal/shelf seas. An important part of such models is the calculation of the flow of energy within the 2-dimensional wavenumber space due to nonlinear wave-wave interaction. Exact computations of the nonlinear energy transfer rate have been performed by e.g. Komen et al. (1984), Resio and Perrie (1989) and Lavrenov (1998), but current forecasting models (e.g. The WAMDI Group, 1988) tend to use less computationally expensive methods, such as the Discrete Interaction Approximation (Hasselmann et al., 1985). More recently, the Reduced Integration Approximation of Lin and Perrie (1999), which approximates the 6-dimensional integrals of the exact formulae by a 1-dimensional integral, may provide a rather good approximation, and is sufficiently fast to be used in forecasting models.

An alternative approach, termed the Local Interaction Approximation by Hasselmann et al. (1985), involves the use of a nonlinear diffusion operator in wavenumber space. In this paper we present an alternative, rather simple diffusion operator expression, which still conserves wave energy, momentum and wave action. It is independent of the details of the dispersion relation, so it can possibly be used for shallow as well as deep water. However, in this paper we only perform explicit calculations for deep water, as in shallow water the assumption of wave action conservation may be violated (Lin and Perrie, 1997).

FORMS OF NONLINEAR SOURCE TERM

We assume that the evolution of the surface gravity wave spectrum (neglecting the effects of currents and varying water depth) can be described by a radiative transfer equation:

$$\begin{aligned}\frac{DF(\mathbf{k};\mathbf{x})}{Dt} &\equiv \frac{\partial F(\mathbf{k};\mathbf{x})}{\partial t} + \nabla_{\mathbf{k}}\,\sigma(\mathbf{k})\cdot\nabla F(\mathbf{k};\mathbf{x}) \\ &= S_{in}(\mathbf{k};\mathbf{x}) + S_{nl}(\mathbf{k};\mathbf{x}) + S_{ds}(\mathbf{k};\mathbf{x})\end{aligned} \tag{1}$$

where $\mathbf{x} \equiv (x,y)$ is the vector of spatial coordinates; $\mathbf{k} \equiv (k_x, k_y) \equiv (k\cos\theta, k\sin\theta)$ is the wavenumber vector; t is time; $\sigma(\mathbf{k})$ is the intrinsic angular frequency, $\nabla_{\mathbf{k}}\sigma$ being the wave group velocity; $\rho g F(\mathbf{k};\mathbf{x})$ is the total wave energy per unit area per unit wavenumber square, ρ being the water density and g the acceleration due to gravity; $S_{in}(\mathbf{k};\mathbf{x})$ is the rate of wave energy input from the wind; $S_{nl}(\mathbf{k};\mathbf{x})$ is the rate of change of wave energy at wavenumber $\mathbf{k}$ and position $\mathbf{x}$ due to nonlinear wave-wave interaction, $-S_{ds}(\mathbf{k};\mathbf{x})$ is the rate of wave energy dissipation by white-capping and other processes; $\nabla \equiv (\partial/\partial x,\ \partial/\partial y)$; and $\nabla_{\mathbf{k}} \equiv (\partial/\partial k_x,\ \partial/\partial k_y)$. Such expressions can also be written for wave momentum:

$$\mathbf{P}(\mathbf{k};\mathbf{x}) = (\mathbf{k}/\sigma)F(\mathbf{k};\mathbf{x}) \tag{2}$$

and wave action:

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