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An Analytic Model for Wave Propagation Across a Crack in an Ice Sheet

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ABSTRACT

By first deriving the appropriate Green's function, a model is developed that allows the interaction of ice-coupled waves with a crack to be studied analytically. Simple formulae for the reflection and transmission coefficients emerge that have not been reported before. The coefficients are found to be strongly dependent on wave period; as period is increased the reflection coefficient drops rapidly to zero, where transmission is perfect, before rising to a low maximum and then decreasing asymptotically to zero again as period is increased further. The mathematical technique employed is amenable to other problems of this type.

INTRODUCTION

Vast areas of the Arctic Ocean are covered with a continuous veneer of sea ice, broken only by cracks, leads and pressure ridges. Such features, which may stretch for tens of kilometres, form when changes in the wind especially cause divergent or convergent stresses to develop in the ice sheet. The ice will crack as it is pulled apart and, if divergent stresses persist, a lead will form that may freeze over. A wind change will then produce a pressure ridge or a shear ridge.

Because waves generated in the open sea are known to penetrate far into the Arctic ice cover, there has been some interest in the possibility of using waves as a tool to remotely sense average ice thickness. Such an idea seems plausible now because, over the last 20 years or so, sustained effort has been put into understanding how waves and sea ice in its many forms interact. (See Squire et al., 1995, for a recent review.) However, the bulk of the theories that have been developed to model waves propagating in sea ice assume that the ice behaves as a uniform plate without flaws.

In this paper, we take a first step towards considering wave propagation in an ice sheet that is not perfect; we consider what effect a single open crack will have on a train of ice-coupled waves. We anticipate that in the near future the analysis will be generalized to deal with wave propagation through an area of heavily cracked sheet ice or, equivalently, a region where the ice has been fractured into floes that are present at high concentration. The shear zone that forms between moving and stationary sea ice in the Arctic basin could be modelled like this.

A single crack in an otherwise perfect ice sheet has been looked at before numerically for a finite depth ocean (Barrett and Squire, 1996), but here we present a full analytical solution for deep water. The method furnishes simple formulae for the reflection and transmission coefficients, R and T, that produce identical results to the Barrett and Squire model without having to solve a complicated matching problem. The formulae, which need only the ice-coupled wave numbers, can easily be evaluated for a given wave period by a few lines of MATLAB code.

To describe the problem mathematically first requires a Green's function to be found for an infinite homogeneous ice sheet floating on the surface of deep water. With this, Green's theorem in the plane is used to write down an equation that directly links the velocity potential $\phi(x, z)$ with the sum of the asymptotic terms at $\pm\infty$ and the terms that arise at the 2 open edges of the cracked ice sheet, recalling that the bending moments and shears must eventually vanish there. At this point, however, they are taken to be continuous only across the crack. Finally, by assuming that the displacements and displacement gradients at these boundaries are finite, the equation may be solved by demanding that the adjoint bending moments and shears also vanish on each crack edge, thereby completing the application of the required number of boundary conditions.

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MATHEMATICAL MODEL

Consider a 1-dimensional sea ice sheet of thickness h and density ρ' floating on deep water of density ρ. The sheet is assumed to be a thin elastic beam of infinite extent with rigidity D. Long-crested, ice-coupled waves of radian frequency ω propagate in the positive x-direction towards a crack in the sheet located at $x = 0$, with the coordinate z assumed to be vertically downwards. The ice-coupled waves are controlled by both the properties of the ice sheet in which they travel and the inertia of the water beneath (Fox and Squire, 1990, 1991, 1994).

In a separate paper Squire and Dixon (2000) consider a related problem, namely waves impinging on an iceberg or thick ice floe trapped in sea ice. Because the first part of the derivation described herein is quite similar, the Green's function and most of the application of Green's theorem being identical, we shall describe the method only briefly here. For a more complete derivation the reader is referred to Squire and Dixon (2000). Assuming then that the velocity potential describing the system is separable and is periodic in time ($e^{-i\omega t}$), the nondimensionalized boundary value problem we seek to solve for $\phi(x, z)$ is:

$$\left.\begin{array}{ll}\nabla^2\phi = 0 & \\ \beta\nabla^4\phi_z + (1-\gamma)\phi_z + \phi = 0, & -\infty < x < 0^-, z = 0 \\ \beta\nabla^4\phi_z + (1-\gamma)\phi_z + \phi = 0, & 0^+ < x < \infty, z = 0 \\ \phi_{zxx}(0^-,0) = \phi_{zxxx}(0^-,0) = \phi_{zxx}(0^+,0) = \phi_{zxxx}(0^+,0) = 0 & \\ \phi_z \to 0, \quad z \to \infty & \end{array}\right\} \quad (1)$$

The nondimensional parameters β and γ are defined $\beta = D\omega^8/\rho g^5$ and $\gamma = \rho' h\omega^2/\rho g$. Suitable radiation conditions must hold at $\pm\infty$ in addition to the equations above.