International Journal of Offshore and Polar Engineering
Vol. 10, No. 2, June 2000 (ISSN 1053-5381)

Statistics for Velocities of Gaussian Waves

Krzysztof Podgórski
Department of Mathematical Sciences, IUPUI, Indianapolis, Indiana, USA

Igor Rychlik and Eva Sjö
Centre for Mathematical Sciences, Lund University, Sweden

ABSTRACT

The sea surface is best modeled as a random field evolving in time. Although a great deal of research has been done on statistical distributions of static characteristics of sea waves, little is known about statistical properties of wave kinematics. By studying distributions for velocities of sea surface motions we are making a step forward in this direction. We extend the approach from the pioneering work of Longuet-Higgins (1957) by taking into consideration the geometry of the sea, as well as its evolution in time. We discuss the following velocities: (1) the ratio of the wave length to the wave period; (2) velocity in the direction of the gradient of the sea surface; (3) velocity of upcrossings and local maxima; (4) velocity of crossing contours; and (5) velocity of wave groups. We derive intensity distributions for these quantities and discuss their interpretation. The results involve generalizations of Rice's formula. They are illustrated by computing these distributions for an example of directional Gaussian sea.

INTRODUCTION

The dispersion relation for a deep-water nonrandom sinusoidal wave:

$$W(x,t) = S\cdot\cos\,(\pi\cdot(x/X + t/T)) \tag{1}$$

propagating in the direction of the x-axis implies that its velocity,defined as $V = -X/T$, satisfies, if $X \geq 0$:

$$V = -\sqrt{g/\pi \cdot X}$$

This velocity can be interpreted as the speed with which the top of the wave moves in space, but also as the speed of a zero-upcrossing or, in fact, of any upcrossing. The velocity is thus constant for the entire wave.

This uniqueness of the wave velocity is no longer valid if one considers the motion of random waves. The wave velocity, or even the wave itself, can be defined in various essentially different ways. In order to reduce at least one source of ambiguity, whenever referring to a wave or an apparent wave, we always assume that it is defined for a spatial profile of the sea by the down-crossing method. The center of the wave is then set at an upcrossing and the half-length X is defined as the distance from the upcrossing to the following (in space) down-crossing (Fig. 1).

Even having the precise notion of a wave (in space), it is still not obvious how to define its velocity, which has to combine space and time properties of the evolving sea. In the case of the sinusoidal wave Eq. 1, the wave length automatically defines the period through the dispersion relation. However, for the random sea, one has to deal with random variability of the sea elevation in both dimensions: in space and in time. Therefore we consider here spatio-temporal waves, which, loosely speaking, are apparent waves together with their histories.

More precisely, we consider $\{W(x,t),\, x \in \mathbb{R},\, t \geq 0\}$ in space and time domain. The wave crests, i.e. the maxima between zero-upcrossings in the x-direction, move with time t. We say that an apparent wave has an extremal crest at an instant τ and a position s if its crest height attains a local maximum in the t-direction at the instant τ. Then an extremal wave is a wave with an extremal crest. In the spatio-temporal domain of W, we mark contours of spatial zero-crossings, contours of spatial local maxima, and points of the extremal spatial crests. An example of a spatio-temporal sea with marked contours and points is given in Fig. 1. Occasionally, we also consider a 2-dimensional random field $W(x,y;t)$ evolving in time. However, because of the complexity of this case, we give a more complete treatment of this case in a future paper. Some further details on the spatio-temporal waves can be found in Podgórski et al. (1999).

The main purpose of this paper is to analyze the kinematics of a random sea by studying statistical properties of various notions of velocity. Their definitions are given below, and we then discuss statistical distributions of these velocities.

The approach presented here has its roots in the following works: Rice (1944, 1945), Longuet-Higgins (1957), Podgórski et al. (2000). See also Podgórski et al. (1999) for more detailed references. The computations of statistical distributions for random waves are based on generalizations of Rice's formula, which can be used under quite general assumptions. For example, one does not need to assume that the underlying process is narrow-banded. Even the assumption that the sea is Gaussian can be weakened, although we do not exploit this possibility in the present work. Generalization of Rice's formula allows to write explicitly, although sometimes in an integral form, statistical distributions of quite arbitrary characteristics of random waves — in our case various wave velocities. The sea state enters into the obtained formulas only through its directional spectrum, which can be given either in a parametric form, or nonparametrically, e.g. estimated

Received October 4, 1999; revised manuscript received by the editors March 9, 2000. The original version (prior to the final revised manuscript) was presented at the Ninth International Offshore and Polar Engineering Conference (ISOPE-99), Brest, France, May 30-June 4, 1999.
KEY WORDS: Directional spectrum, Gaussian sea, Rice's formula, velocity of local maxima, velocity of level crossings, level crossing contours, wave groups.